Using Dynamic Geometry Software to Investigate Midpoint Quadrilateral

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Abstract

This article describes the use of dynamic geometry software by pre-service secondary school teachers in a problem solving investigation. The dynamic nature of Geometer Sketchpad helped students navigate through the investigative process by helping them build and prove/disprove conjectures. Students investigated two issues: 1) the midpoint quadrilateral created by connecting the midpoints of the sides of a given quadrilateral; and 2) the relationship between the area of the midpoint quadrilateral and the given quadrilateral. This activity suggests that students benefit from mathematical exploration by giving them the opportunity to make and test conjectures.

Introduction

The NCTM standards [7] argue that students should be provided with the opportunity to investigate mathematical problems as an aid to understanding mathematical ideas. Ponte [8] points out "a mathematical investigation stresses mathematical processes such as searching regularities, formulating, testing, justifying and proving conjectures, reflecting, and generalizing (p. 54)". To engage students in mathematical investigations, educators should provide students with tools that enhance the process. Dynamic geometry software such as the Geometer Sketchpad (GSP) [6] is a tool to reach that goal. As research studies (e.g., [1], [2], [4]) indicate, dynamic geometry software can be a cognitive tool to help students develop problem-solving abilities if used effectively. They identified different purposes for which students used dragging, the main feature of the dynamic geometry software, and different purposes for which students used measures. These purposes appeared to be influenced by students' mathematical understandings that were reflected in how they reasoned about the physical representations, the types of abstractions they made, and the reactive or proactive strategies employed. These issues are key to the investigation reported herein.

This article reports on the following class activity in the form of an investigation that was implemented by us in our college geometry class for high school presevice teachers using the Geometer Sketchpad:

> Given any quadrilateral, construct a midpoint on each side. Connect each consecutive midpoint with a segment. What are the properties of the shape formed by joining the midpoints? Does the resulting shape depend on the type of

quadrilateral (e.g. convex & concave)? How does the area of the midpoint quadrilateral compare to the area of the original quadrilateral? This activity was adapted from the Intermath website [5].

Part I. The investigation on the shape of the midpoint quadrilateral

We started the activity by asking students to construct any quadrilateral and its midpoint quadrilateral using GSP. They worked in a group of three with a computer. This was a rather straightforward task for most of them. One of the students did the construction by first constructing four points A, B, C and D, using the GSP selection tool to select these points in clockwise order, and then constructed segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$, which formed a quadrilateral (Figure 1a). Then the student selected the four segments and constructed midpoints F, G, H, and E on the segments respectively. The student then constructed segments $\overline{EF}$, $\overline{FG}$, $\overline{GH}$ and $\overline{HE}$ to form the midpoint quadrilateral as shown in Figure 1b. Other students constructed their midpoint quadrilaterals in a similar way.

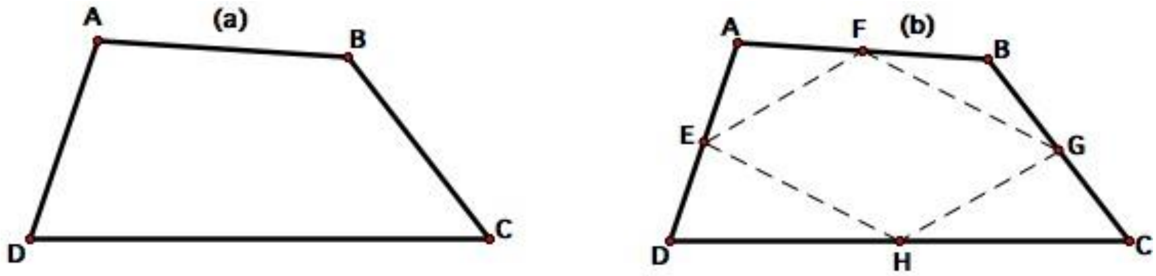

Figure 1: A quadrilateral and its midpoint quadrilateral

The students then investigated the shapes formed – quadrilateral EFGH. Using the dynamic movement feature of GSP, they clicked one of the vertices (A, B, C or D) of the original quadrilateral and dragged it around to see what changes they could notice. In so doing, one group reported to the rest of the class that the original quadrilateral could be one of the following: 1) convex quadrilateral[1] (Figure 2a), 2) concave quadrilateral[2] (Figure 2b) and 3) a type of quadrilateral that was new to them – crossed quadrilateral[3] (Figure 2c). (Convex and concave quadrilaterals are two types of simple quadrilateral whereas crossed quadrilateral is a non-simple quadrilateral.) The students in one of the groups pointed out that quadrilateral EFGH – the shape formed by joining the midpoints of the sides of the original quadrilateral (any of the three types) seemed to have two pairs of parallel sides and hence seemed to be a parallelogram. While the rest of the class agreed with this finding, some students did not arrive at the result on their own.

[1] Convex quadrilateral is any quadrilateral with no diagonal falling outside the figure.
[2] **C**oncave quadrilateral is any quadrilateral with one diagonal falling outside the figure.
[3] Crossed quadrilateral is any quadrilateral with both diagonals falling outside the figure.

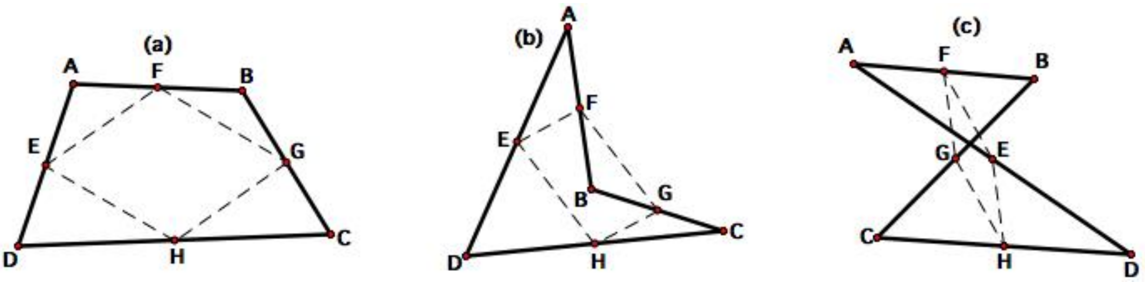

Figure 2: The three types of the original quadrilateral

Since this was an open investigation, not all students used the measurement feature of GSP fully. For those who took the measurement of the sides and angles of the midpoint quadrilateral, they wanted to know what would happen as they drag one of the vertices of the original quadrilateral to change its shape and size. Based on what they observed (see Figure 3), one of the students stated, "No matter whether the (original) quadrilateral is concave, convex or crossed, the opposite sides (of the midpoint quadrilateral) always have the same measurement and the opposite angles have the same measurement too. No matter how many times one changes the (original) quadrilateral, the midpoint quadrilateral always has those same characteristics." Other students agreed with him. We asked the class to write in a piece of paper what type of quadrilateral is the midpoint quadrilateral. The answers from most of the students indicated that it was a parallelogram, while the answers from a few of them were rectangle or square. We suggested that the students drag the original quadrilateral around and observe the changes of the angle measures again. This time they noticed that the angles did not remain right angles. They then ruled out the possibility that the quadrilateral was always a rectangle or a square. Thus, the students conjectured that the midpoint quadrilateral is a parallelogram, and this fact holds for all three types of quadrilateral. Not all students arrived at this consensus independently but rather through discussions in individual groups and whole class discussion.

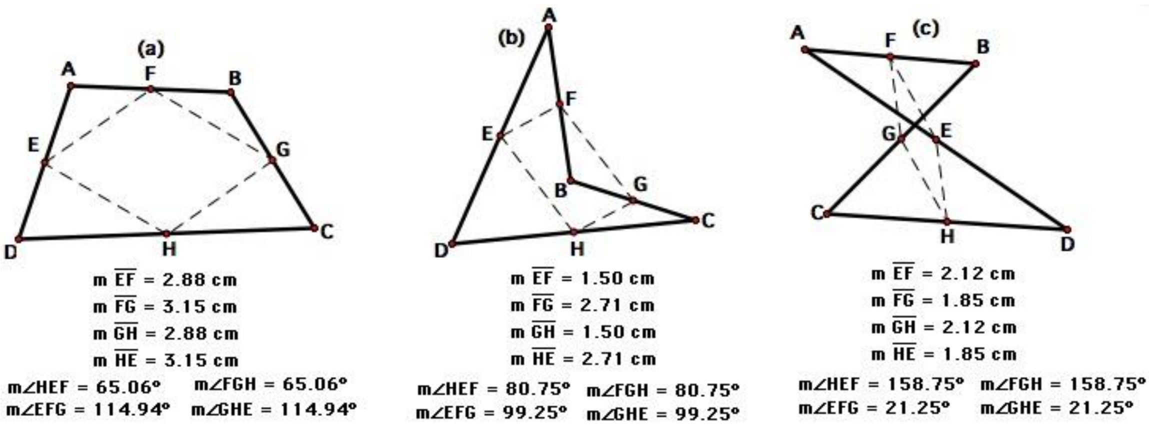

Figure 3: Side length and angle measurements of the midpoint quadrilateral

We now asked the students to try to prove their conjecture. Some of the students thought they had already done the proof by stating that their reasoning was not based on one case but on all possible cases with the GSP dynamic movement feature. We challenged them to examine their reasoning by asking, "You indicated that the opposite sides of quadrilateral EFGH always have the same length and its opposite angles are always congruent. Did these properties come from your measurement using GSP, or from your logical reasoning?" Through serious discussion, the students realized that the measurement is very important in the investigation process, but it is still different from a mathematical proof that begins with a truth and proceeds by logical steps to a conclusion, which then must be true. They then worked hard on constructing a proof as a group in class and were given ample time as they shared ideas among themselves.

The following is the proof that one student came up with for the convex quadrilateral case (with minor help from us):

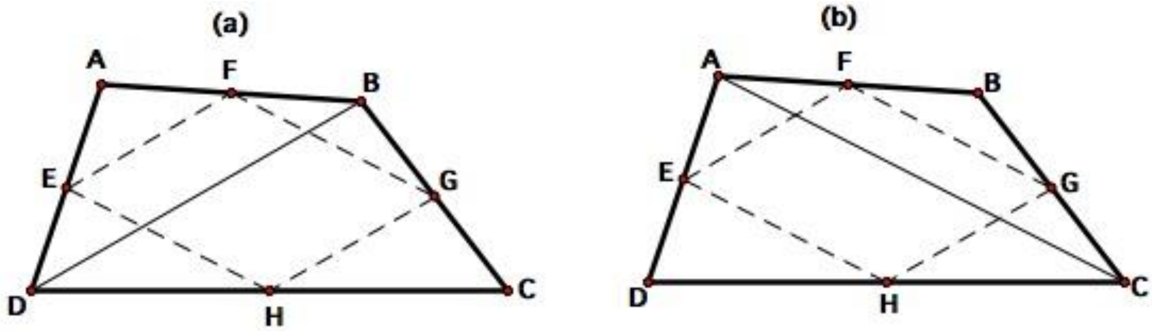

Figure 4: The midpoint quadrilateral is a parallelogram

> Start with figure 4a and construct segment $\overline{DB}$ (diagonal of quadrilateral ABCD). In ΔDAB, $\overline{EF}$ is a mid-segment? By the Mid-segment Theorem[4], $\overline{EF}$ // $\overline{DB}$ and m ($\overline{EF}$) = (1/2)*m($\overline{DB}$). In ΔDCB, $\overline{HG}$ is a mid-segment, and again by the Mid-segment Theorem, $\overline{HG}$ // $\overline{DB}$ and m($\overline{HG}$) = (1/2)*m($\overline{DB}$). By transitivity, $\overline{EF}$ // $\overline{HG}$ and m($\overline{EF}$) = m($\overline{HG}$). Therefore the midpoint quadrilateral is a parallelogram (as it has a pair of opposite sides being both parallel and congruent).

While two groups of the students used the same method, the rest of the class worked out their proofs using slightly different ways. Instead of proving one pair of opposite sides parallel and congruent, they either proved two pairs of opposite sides being parallel, or proved two pairs of opposite sides being congruent. For either of the two ways, two diagonals (see Figure 4a and Figure 4b) rather than just one needed to be constructed. By sharing the proofs, the students not only developed multiple ways of proof, but also deepened their understanding of the properties and conditions of a parallelogram.
The discussion on the proof for the midpoint quadrilateral of the convex quadrilateral made sense to students in terms of visualizing mid-segments as the diagonal(s) are drawn. But it was a different story for the situations with the concave quadrilateral case and crossed quadrilateral case. It was difficult for some students to see where the diagonals of the quadrilateral are and how to use them to prove that the midpoint quadrilateral is a parallelogram.

[4] This theorem states that the segment connecting the midpoints of two sides of a triangle (mid-segment) is parallel to and one half as long as the third side.

One student suggested to start with a convex quadrilateral and its diagonals first, and then use the dynamic features of GSP to move one of the vertices inward to form the concave quadrilateral case (Figure 5), or even further to form the crossed quadrilateral case (Figure 6). This was a good idea to help the rest of the class to visualize the diagonals ($\overline{DB}$ and $\overline{AC}$), and to make the proof process almost identical to that of the convex quadrilateral case. Thus, students came up with their proofs without difficulty in their written work, which was assessed in class by the instructors through discussing and evaluating the proofs with individual students.

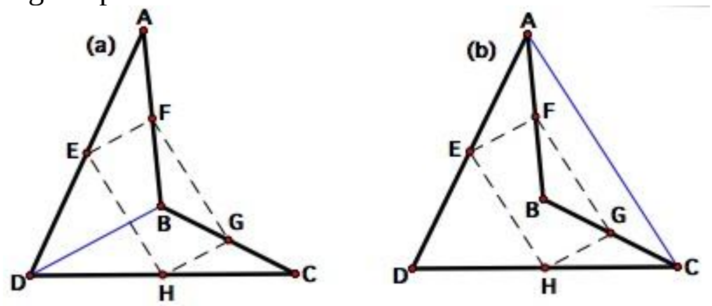

Figure 5: The midpoint quadrilateral of a concave quadrilateral

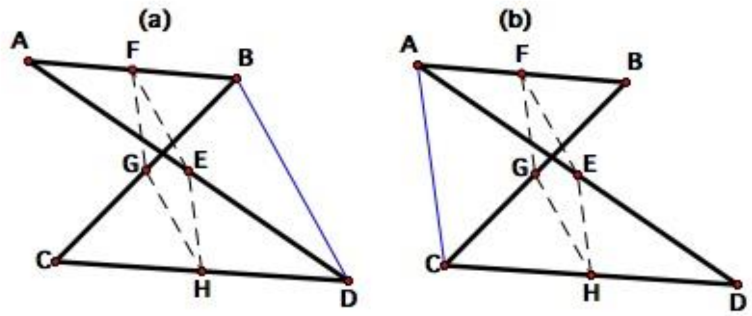

Figure 6: The midpoint quadrilateral of a crossed quadrilateral

Part II. The investigation on the area of the midpoint quadrilateral

After the investigation on the shape of the midpoint quadrilateral, the students continued to explore the relationship between the area of the midpoint quadrilateral (EFGH) and that of the original quadrilateral (ABCD). Using GSP, they first chose Quadrilateral Interior from the construction menu to give quadrilateral ABCD and quadrilateral EFGH different colors. Then they measured the colored areas of both quadrilaterals. With a hint from us, they found the ratio of the two areas and noticed an interesting fact: The area of the midpoint quadrilateral EFGH is half the area of the original quadrilateral ABCD. Using the dynamic feature of GSP, the students dragged one of the vertices of quadrilateral ABCD around to observe any possible changes in terms of area measures and the ratio. They noted that the finding *(the area of quadrilateral EFGH is half the area of quadrilateral ABCD)* holds for all types of quadrilateral (convex, concave, and crossed) (Figure 7). Of course, for the crossed quadrilateral, since it is not a simple quadrilateral, its area is not the sum of the areas of the two triangles (ΔABK and ΔCDK, where K is the intersection point of sides $\overline{BC}$ and $\overline{DA}$) seen in Figure 7c, but the difference of those areas, or in other words, the sum of signed areas of the triangles. In this case, we regard the area of a triangle as being positive or negative according as its vertices are named in counterclockwise or clockwise order, as indicated by de

Villiers in [3]. Specifically here, Area(ΔBAK) is positive, but Area(ΔABK) is negative. Therefore, Area(crossed quadrilateral ABCD in Figure 7c) = Area(ΔCDK) - Area(ΔBAK), or Area(ΔCDK) + Area(ΔABK).

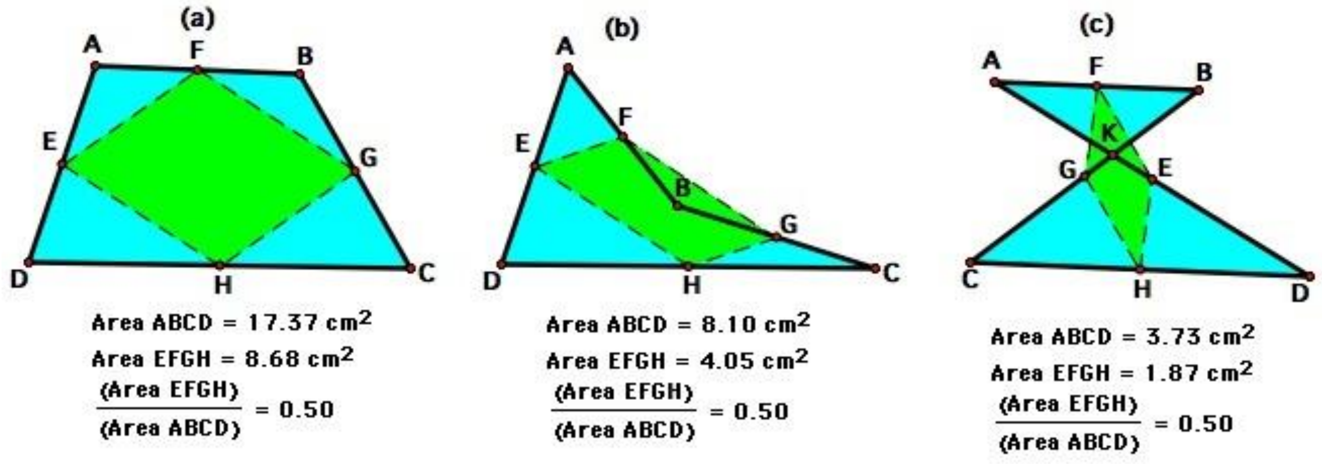

Figure 7: The areas of quadrilateral ABCD and its midpoint quadrilateral

The proof of this conjecture was not easy for most of the students. Except in the special cases (e.g., quadrilateral ABCD is either a rectangle or a square), the students have difficulties in coming up with the proof. To help students get started, we discussed the following idea:

> Lemma 1: In an arbitrary triangle ABC, if D is the midpoint of AB, and E is the midpoint of AC, then $Area(\Delta ADE) = \frac{1}{4} Area(\Delta ABC)$.

In doing this, we used a GSP sketch (Figure 8) to demonstrate the lemma and students noticed that the ratio of the area of ΔADE to the area of ΔABC was always 1:4 no matter how the figure changed in shape. The dynamic feature of GSP enabled students to click and drag a vertex and observe that the ratio of the two areas was always 1:4.

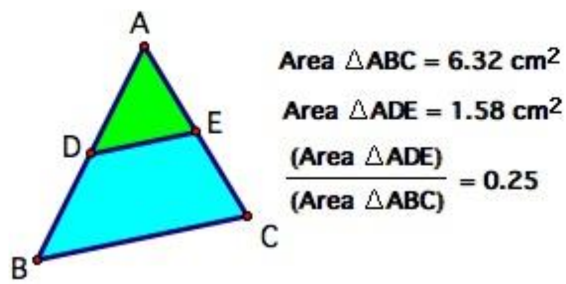

Figure 8: The ratio of the area of ΔADE to the area of ΔABC is always 1:4

We asked the students how one can prove the lemma and they gave suggestions such as finding the heights of the two triangles, and considering trigonometric ratios. With minimal help, one student volunteered to present his proof to the rest of the class as follows:

In Figure 8, $Area(\Delta ADE) = \frac{1}{2} AD \times AE \times \sin(\angle A)$, $Area(\Delta ABC) = \frac{1}{2} AB \times AC \times \sin(\angle A)$

Since D is the midpoint of $\overline{AB}$, and E is the midpoint of $\overline{AC}$, $AD = \frac{1}{2} AB$, and $AE = \frac{1}{2} AC$

Therefore, $\frac{Area(\Delta ADE)}{Area(\Delta ABC)} = \frac{\frac{1}{2} AD \times AE \times \sin(\angle A)}{\frac{1}{2} AB \times AC \times \sin(\angle A)} = \frac{AD \times AE}{AB \times AC} =$

$\frac{\frac{1}{2} AB \times \frac{1}{2} AC}{AB \times AC} = \frac{1}{4}$. That is,

$Area(\Delta ADE) = \frac{1}{4} Area(\Delta ABC)$.

Using Lemma 1, the students discussed ways of proving the three cases (convex, concave and crossed quadrilaterals) in Figure 7. With some help (e.g., the suggestion of using backward reasoning) from us, the students developed the following proofs:

Case 1: Convex Quadrilateral ABCD

Proof: Construct diagonal $\overline{AC}$. Since F is the midpoint of $\overline{AB}$ and G is the midpoint of $\overline{BC}$ then $Area(\Delta BFG) = \frac{1}{4} Area(\Delta BAC)$ by Lemma 1 … (1)

Use the same reasoning, we can prove the following:

$Area(\Delta DEH) = \frac{1}{4} Area(\Delta DAC)$ … (2),

$Area(\Delta AEF) = \frac{1}{4} Area(\Delta ADB)$ … (3), and

$Area(\Delta CHG) = \frac{1}{4} Area(\Delta CDB)$ … (4)

Proofs of 3 and 4 may require constructing diagonal $\overline{BD}$ (Figure 9).

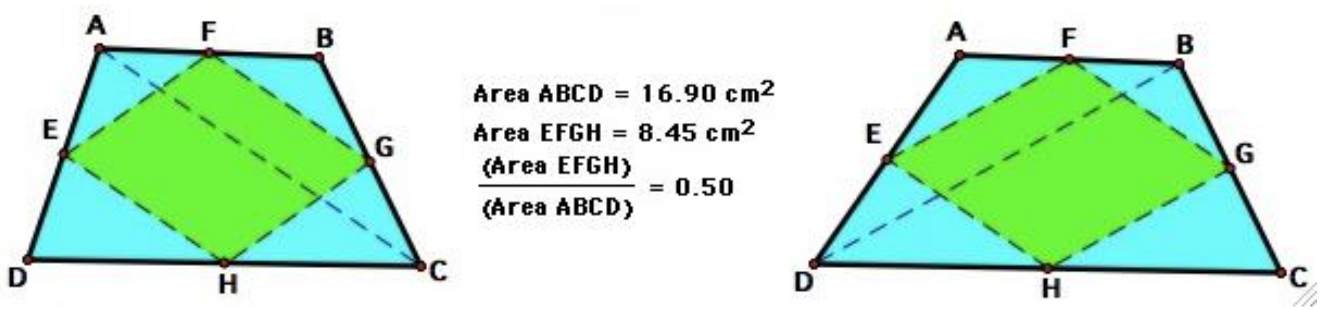

Figure 9: Area of the midpoint quadrilateral of a convex quadrilateral

If we add (1), (2), (3), and (4), we have
$Area(\Delta BFG) + Area(\Delta DEH) + Area(\Delta AEF) + Area(\Delta CHG)$
$= \frac{1}{4}Area(\Delta BAC) + \frac{1}{4}Area(\Delta DAC) + \frac{1}{4}Area(\Delta ADB) + \frac{1}{4}Area(\Delta CDB)$
$= \frac{1}{4}\left[Area(\Delta BAC) + Area(\Delta DAC) + Area(\Delta ADB) + Area(\Delta CDB)\right]$
$= \frac{1}{4}\left[Area(ABCD) + Area(ABCD)\right] = \frac{1}{4}\left[2 \times Area(ABCD)\right]$
$= \frac{1}{2}Area(ABCD).$
Therefore, Area (EFGH) = Area (ABCD) –
$\left[Area(\Delta BFG) + Area(\Delta DEH) + Area(\Delta AEF) + Area(\Delta CHG)\right]$
$= Area(ABCD) - \frac{1}{2}Area(ABCD) = \frac{1}{2}Area(ABCD)$. That is, the area of the midpoint quadrilateral of a convex quadrilateral is a half the area of the quadrilateral.

Case 2: Concave Quadrilateral ABCD

Poof: Construct diagonals $\overline{AC}$ and $\overline{BD}$ (Figure 10).
Since E is the midpoint of $\overline{AD}$ and F is the midpoint of $\overline{AB}$, by Lemma 1,
$Area(\Delta AEF) = \frac{1}{4}Area(\Delta ADB)$ … (1).
Use the same reasoning, we can prove:
$Area(\Delta CHG) = \frac{1}{4}Area(\Delta CDB)$ … (2),
$Area(\Delta DEH) = \frac{1}{4}Area(\Delta DAC)$ … (3), and
$Area(\Delta BFG) = \frac{1}{4}Area(\Delta BAC)$ … (4).

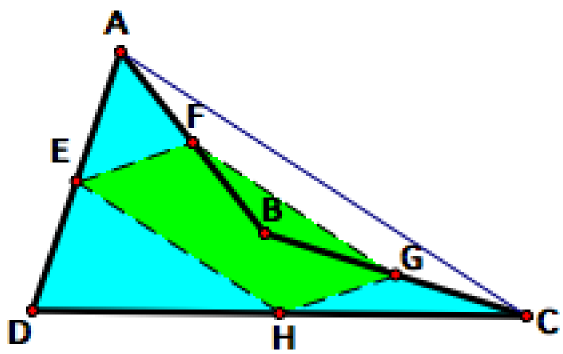

Figure 10: Area of the midpoint quadrilateral of a concave quadrilateral

(1) + (2) yields
$Area(\Delta AEF) + Area(\Delta CHG) = \frac{1}{4}Area(\Delta ADB) + \frac{1}{4}Area(\Delta CDB) = \frac{1}{4}Area(ABCD)$… (5)

(3) – (4) yields $Area(\Delta DEH) - Area(\Delta BFG) = \frac{1}{4} Area(\Delta DAC) - \frac{1}{4} Area(\Delta BAC)$ =

$\frac{1}{4}\left[Area(\Delta DAC) - Area(\Delta BAC)\right]$ = $\frac{1}{4} Area(ABCD)$ … (6)

(5) + (6) yields

$Area(\Delta AEF) + Area(\Delta CHG) + Area(\Delta DEH) - Area(\Delta BFG)$ =

$\frac{1}{4} Area(ABCD) + \frac{1}{4} Area(ABCD) = \frac{1}{2} Area(ABCD)$ =>

Area (EFBGH) =Area (ABCD) – $\left[Area(\Delta AEF) + Area(\Delta CHG) + Area(\Delta DEH)\right]$

= Area (ABCD) – $\frac{1}{2} Area(ABCD) - Area(\Delta BFG)$.

Therefore, we have

Area (EFBGH) + $Area(\Delta BFG) = \frac{1}{2} Area(ABCD)$ =>

$Area(EFGH) = \frac{1}{2} Area(ABCD)$. That is, the area of the midpoint quadrilateral of a concave quadrilateral is a half the area of the quadrilateral.

Case 3: Crossed Quadrilateral

Proof: For the sake of simplicity, in the following figure (Figure 11), we use labels to express areas (for example, use y1 to express the area of ΔCHG). K is not one of the vertices of quadrilateral ABCD. However, when we look at this crossed quadrilateral as two triangles, K is the shared vertex of them (ΔBAK and ΔCDK). This will be used in the proof.

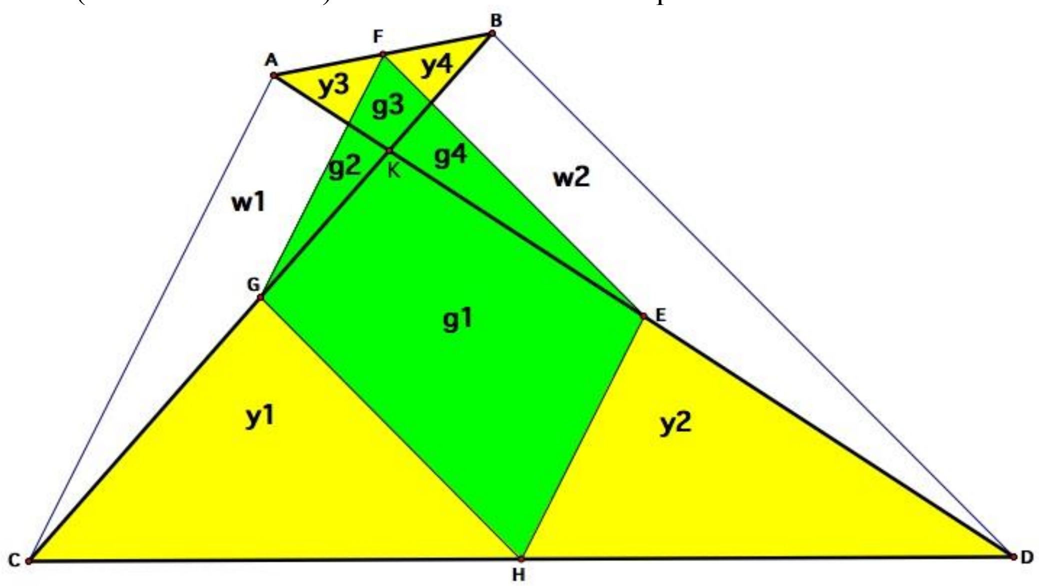

Figure 11: Area of the midpoint quadrilateral of a crossed quadrilateral

From Lemma1, y1 = (1/4)*(y1 + y2 + g1 + g4 + w2) => 3y1 = y2 + g1 + g4 + w2 … (1)

Use the same reasoning, we can prove:

3(y3 + g3 + g4) = y4 + w2 => 3y3 = y4 + w2 – 3g3 – 3g4 … (2)

$3y2 = y1 + g1 + g2 + w1$ … (3)
$3(y4 + g3 + g2) = y3 + w1 \Rightarrow 3y4 = y3 + w1 – 3g3 – 3g2$ … (4)

(1) – (2): $3(y1 – y3) = y2 – y4 + g1 + 3g3 + 4g4$ … (5)
(3) – (4): $3(y2 – y4) = y1 – y3 + g1 + 3g3 + 4g2$ … (6)

(5) + (6): $3(y1 – y3) + 3(y2 – y4) = (y1 – y3) + (y2 – y4) + 2g1 + 4g2 + 6g3 + 4g4$
$\Rightarrow (y1 – y3) + (y2 – y4) = g1 + 2g2 + 3g3 +2g4$
$\Rightarrow (y1 + y2 + g1) – (y3 + y4 + g3) = 2(g1 + g2 + g3 + g4)$,
which is Area(ΔCDK) - Area(ΔBAK) = 2*Area(EFGH).

According to the explanation for the area of a crossed quadrilateral described above, the area of midpoint quadrilateral EFGH is half the area of the original quadrilateral ABCD.

The proofs (for Cases 1, 2, and 3) described above were done by one group of students with some help from us who finally presented the proofs to the rest of the class. These proofs were developed separately. In order to help students see the connections among these cases, at the end of the investigations described in this article, we introduced them to Bretschneider's Formula (http://home.att.net/~numericana/answer/formula.htm#bretschneider). Bretschneider's Formula does hold for convex and concave quadrilaterals, and it's also good for (the signed areas of) crossed quadrilaterals. Expressed as $(4A)^2 = 4p^2q^2 – (a^2 – b^2 + c^2 – d^2)^2$, the formula gives the simplest way to express the area (A) of a quadrilateral in terms of its sides (a, b, c, d) and diagonals (p, q) without any restrictions.

Conclusion

Our experiences working with the pre-service secondary school teachers who participated in this activity using GSP indicate that pre-service teachers can benefit from mathematical explorations and investigations with dynamic geometry software as a tool. The activities shown in the article indicate that giving students the opportunity to pose and prove/disprove conjectures with the aid of GSP can be a great learning tool. The three cases of quadrilateral (concave, convex and crossed) are easily investigated by virtue of the existence of those features of GSP. (Please also see [3] for a thorough discussion in this aspect.] The dynamic features enable one to change convex quadrilateral to concave and then to crossed and back to convex by simply dragging one point of the quadrilateral. With the aid of measurement feature of GSP, students can make conjectures in terms of what they see as the quadrilateral changes from one case to another.

Due to the dynamics and measurement feature of GSP, it can construct visual representations easily and accurately as opposed to static figures that can be done with paper and pencil. These features in GSP make it possible for students to concentrate more on the problem than just struggling with a visual representation to model a given situation.

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Supplementary Files

[9] GSP files for Figures 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11.